Proposition 1:

The regularized determinant of Hessian Q of CS-functional at flat connection α satisfies

$$\frac{\sqrt{\det d_\alpha^* d_\alpha}}{\sqrt{|\det Q|}} = T_\alpha^{1/2}$$

where T_α is the Ray-Singer torsion

$$T_\alpha(M) = \frac{(\det \Delta_\alpha^0)^{3/2}}{(\det \Delta_\alpha^1)^{1/2}}$$

Proof:

By the identity $P^2 = \Delta_\alpha^0 \oplus \Delta_\alpha^1$ for

$$P = \begin{pmatrix} 0 & -d_\alpha^* & 0 & 0 \\ -d_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \qquad (*)$$

we have $\det P^2 = \det \Delta_\alpha^0 \oplus \det \Delta_\alpha^1$. Using $(*)$, we see

$$|\det P| = \underbrace{\det(d_\alpha^* d_\alpha)}_{=\det \Delta_\alpha^0} |\det Q|$$

which gives

$$|\det Q| = \frac{(\det \Delta_\alpha^1)^{1/2}}{(\det \Delta_\alpha^0)^{1/2}}$$

Using $\Delta_\alpha^0 = d_\alpha^* d_\alpha \longrightarrow$ claim follows $\square$

The phase of the determinant:

Recall that

$$\int_{-\infty}^{\infty} e^{i\lambda x^2}\, dx = \sqrt{\frac{\pi}{|\lambda|}}\, e^{\frac{\pi i}{4}\frac{\lambda}{|\lambda|}}$$

$\rightarrow$ the phase is proportional to $\sum_i \operatorname{sign}\lambda_i$

In the case of Chern-Simons theory, the correct generalization is the "eta invariant":

$$\eta(\alpha) = \frac{1}{2} \lim_{s \to 0} \sum_i \operatorname{sign}\lambda_i |\lambda_i|^{-s}$$

$$\rightarrow \quad \frac{1}{\sqrt{\det Q}} = \frac{1}{\sqrt{|\det Q|}} \cdot \exp\left(\frac{i\pi}{2}\eta(\alpha)\right)$$

The Atiyah-Patodi-Singer index theorem then gives

$$\frac{i\pi}{2}\left(\eta(\alpha) - \eta(0)\right) = 2\pi i \underbrace{h^\vee(G)}_{=2 \text{ for } G = SU(2)} CS(\alpha)$$

Let us now put all steps together and compute the asymptotic behaviour of $Z_k(M)$:

$$Z_k(M) = \int \exp\left(2\pi\sqrt{-1}\, k\, CS(A)\right) \mathcal{D}A$$

as $k \to \infty$

Recall that $CS(A)$ is degenerated along the orbit of the gauge group

$\longrightarrow \sqrt{\det d_\alpha^* d_\alpha}$ is interpreted as the volume of the gauge group

Thus we obtain:

$$Z_K(M) \sim_{K \to \infty} e^{i\pi \eta(0)/2} \cdot \sum_\alpha \sqrt{T_\alpha(M)}\, e^{2\pi i (K + h^\vee) CS(\alpha)}$$

$CS(\alpha)$ and $T_\alpha(M)$ are topological invariants <u>but</u> $\eta(0)$ is <u>not</u>!

<u>Trivialization of the tangent bundle:</u>

$\eta(0)$ is the η invariant of the Q-operator coupled to

1) some metric g on M
2) trivial gauge field $A=0$

Let $d = \dim G$

$2) \longrightarrow \eta(0) = d \cdot \eta_{grav}$ (grav here = metric dep.)

$$\longrightarrow \Lambda = \exp\left(\frac{id\pi}{2} \cdot \eta_{grav}\right)$$

Define gravitational Chern-Simons term:

$$CS(g) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega\right)$$

where ω is the Levi-Civita connection on the spin bundle of M.

$\longrightarrow$ requires trivialization of tangent bundle

The Atyiah-Patodi-Singer index theorem says:

$$\frac{1}{2} \eta_{grav} + \frac{1}{12} \cdot CS(g)$$

is a topological invariant of M (but depends on framing)

$\longrightarrow$ define

$$Z_K \sim e^{i\pi d\left(\frac{\eta_{grav}}{2} + \frac{1}{12} CS(g)\right)} \cdot \sum_{\alpha} \sqrt{T_\alpha}\, e^{2\pi i (K+h^\vee) CS(\alpha)}$$

$\longrightarrow$ topological invariant

If the framing is shifed by s units, Z_K transforms as

$$Z_K \longrightarrow Z_K \cdot \exp\left(2\pi i s \frac{d}{24}\right)$$

<u>Note:</u> $\lim_{K \to \infty} c = d$

§12. Chern-Simons perturbative invariants

We start with $G = U(1)$. Let $L = K_1 \cup K_2$ be an oriented framed link with two comp. in $\mathbb{R}^3$. Define

$$P := \text{Principal } U(1) \text{ bundle on } \mathbb{R}^3$$

$$\mathcal{A}_{\mathbb{R}^3} := \text{space of connections on } P$$

$\longrightarrow$ Chern-Simons partition function

$$Z_K = \int_{\mathcal{A}_{\mathbb{R}^3}} \exp\left(K \frac{\sqrt{-1}}{4\pi} \int_{\mathbb{R}^3} A \wedge dA + \sqrt{-1} \int_{K_1} A + \sqrt{-1} \int_{K_2} A\right) \mathcal{D}A$$

$A \mapsto A \wedge dA$ defines a quadratic form on $\mathcal{A}_{\mathbb{R}^3}$

Finite-dimensional analogy:

$$Q(x_1, \dots, x_n) = \frac{1}{2} \sum_{i,j} \lambda_{ij} x_i x_j$$

$$\longrightarrow \int_{\mathbb{R}^n} e^{\sqrt{-1}\left(Q(x_1, \dots, x_n) + \sum_{j=1}^{n} \mu_j x_j\right)} dx_1 \dots dx_n$$

$$\sim e^{-\sqrt{-1} \sum_{i,j} \lambda^{ij} \mu_i \mu_j} \qquad (*)$$

where (λ^{ij}) is inverse matrix of (λ_{ij})

In the case of the operator d, the inverse is an integral operator:

$$d\, L(\vec{x}, \vec{y}) = \delta^{(3)}(\vec{x}, \vec{y}) \qquad (**)$$

$$(\hat{L}\,\varphi)(\vec{x}) = \int_{y \in \mathbb{R}^3} L(\vec{x}, \vec{y}) \wedge \varphi(\vec{y}), \quad \varphi \text{ 1-form}$$

$\longrightarrow$ solution is given by the "Green form":

For $\vec{x} \in \mathbb{R}^3 \setminus \{\vec{0}\}$ we put

$$\omega(\vec{x}) = \frac{1}{4\pi} \frac{x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2}{\|\vec{x}\|^3}$$

$$\int_{S^2} \omega = 1$$

Then the solution of $(**)$ is given by

$$L(\vec{x}, \vec{y}) = \omega(\vec{x}, \vec{y})$$

$\longrightarrow$ analog of $(*)$ is:

$$Z_K \sim \exp\left(\frac{\sqrt{-1}\,\pi}{K} \sum_{i,j} I(K_i, K_j) \right)$$

where $I(K_i, K_j)$, $1 \leq i, j \leq 2$, is given by

$$I(K_i, K_j) = \int_{\vec{x} \in K_i, \vec{y} \in K_j} \omega(\vec{x} - \vec{y})$$

if $i \neq j$.

For $i=j$,

$$I(K_i, K_i) = \int_{\vec{x}\in K_i,\, \vec{y}\in K_i'} \omega(\vec{x}-\vec{y})$$

where K_i' is a curve on the boundary of a tubular neighborhood of K_i'.

<u>Now let us proceed to $G = SU(2)$.</u>

Finite-dim. analogy:

$$Z_K = \int_{\mathbb{R}^n} e^{\sqrt{-1}\, K f(x_1, \dots, x_n)}\, dx_1 \dots dx_n$$

→ restric to case:

$$f(x_1, \dots, x_n) = Q(x_1, \dots, x_n) + \sum_{i,j} \lambda_{ijk} x_i x_j x_k$$

non-degenerate quadratic form

Change of variables gives

$$Z_K = K^{-n/2} \int_{\mathbb{R}^n} e^{\sqrt{-1}\, Q(x_1, \dots, x_n)} \times \sum_{m=0}^{\infty} \frac{(\sqrt{-1})^m}{m!\, K^{m/2}} \left(\sum_{i,j,k} \lambda_{ijk} x_i x_j x_k \right)^m dx_1 \dots dx_n .$$

→ obtain asymptotic expansion for $K \to \infty$

We compute

$$\int_{\mathbb{R}^n} e^{\sqrt{-1}\,Q(x_1,\dots,x_n)}\left(\sum_{i,j,k}\lambda_{ijk}\,x^i x^j x^k\right)^m dx_1\dots dx_n$$

$$= \left[\left(\sum_{i,j,k}\lambda_{ijk}\,D_i D_j D_k\right)^m \int_{\mathbb{R}^n} e^{\sqrt{-1}\left(Q(x_1,\dots x_n)+\sum_k J_k x_k\right)} dx_1\dots dx_n\right]_{J=0}$$

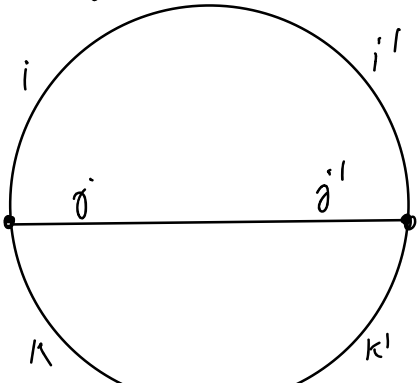
where $D_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial J_j}$

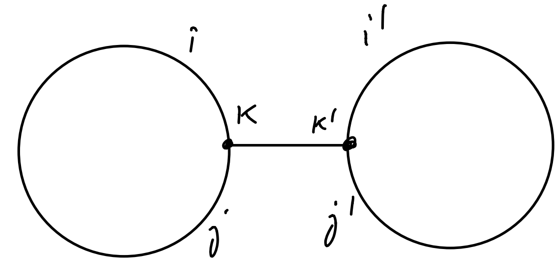
complete square

$$= \left[\left(\sum_{i,j,k}\lambda_{ijk}\,D_i D_j D_k\right)^m e^{-\sqrt{-1}\frac{1}{2}\sum_{i,j}\lambda^{ij}J_i J_j}\right]_{J=0}$$

In the case $m=2$, we get

$$\sum_{i,j,k,i',j',k'}\lambda_{ijk}\lambda_{i'j'k'}\lambda^{ii'}\lambda^{jj'}\lambda^{kk'} + \sum_{ijki'j'k'}\lambda_{ijk}\lambda_{i'j'k'}\lambda^{ij}\lambda^{kk'}\lambda^{i'j'}$$





$\rightarrow$ Feynman diagram expansion